[9]Gebhart, B., and Mollendorf, J. C., "A New Density Relation for Pure and Saline Water," *Deep-Sea Research*, Vol. 24, No. 9, 1977, pp. 831–848.

[10]Hwang, Y., "The Stability of Some Steady-States and the Effect of Motion Pressure in Vertical Natural Convection Flows in Cold Water," Ph.D. Dissertation, State University of New York, Buffalo, NY, 1984.

Implicit Solution Method for a Cumulative Variable Formulation to Radiative/Conductive Transport

J. I. Frankel*
University of Tennessee,
Knoxville, Tennessee 37996-2210

I. Introduction

In a recent article,[1] a new formulation was presented for transient, one-dimensional conductive and radiative transport in a gray, plane-parallel medium. The introduction of cumulative variables[2,3] and the subsequent use of a concept offered by Kumar and Sloan[4] permitted the complicated integro-differential formulation to be readily and accurately resolved by orthogonal collocation. Several questions naturally arise with regard to computation, and thus, these issues deserve attention early in the developmental stages. Germane to this Note is the unfolding of an efficient numerical method for resolving the unknown expansion coefficients[1] presented by the collocation approach.

This Note presents an implicit numerical method for solving the time-varying expansion coefficients developed by Frankel.[1] Thus, the focus is directed toward the presentation of an efficient and accurate block-by-block method[5–7] for solving a system of nonlinear Volterra integral equations of the second kind. Finally, some new numerical results are presented when the single-scattering albedo is zero and unity.

II. Implicit Formulation and Analysis

For the sake of uniformity, the nomenclature used in this Note is identical to that presented in Ref. 1. Additionally, since this Note addresses only the computational methodology associated with resolving the expansion coefficients expressed by Frankel,[1] only details concerning these functions are presented. Reconstruction of the needed solutions is shown by Frankel.

The coupled, differential/algebraic matrix system of equations for the expansion coefficients of the three cumulative variables $\{\Psi_k^N(\eta, \xi)\}_{k=1}^3$ described by Frankel are

$$B\frac{d\bar{a}}{d\xi}(\xi) = \bar{g}(\xi) \tag{1a}$$

$$B\frac{d\bar{b}}{d\xi}(\xi) = \bar{h}(\xi), \qquad \xi > 0 \tag{1b}$$

$$A\bar{c}(\xi) = \bar{f}(\xi), \qquad \xi \geq 0 \tag{1c}$$

where $\bar{a}(\xi) = [a_0^N(\xi), a_1^N(\xi), \ldots, a_N^N(\xi)]^T$, $\bar{b}(\xi) = [b_0^N(\xi), b_1^N(\xi), \ldots, b_N^N(\xi)]^T$, and $\bar{c}(\xi) = [c_0^N(\xi), c_1^N(\xi), \ldots, c_N^N(\xi)]^T$. The elements of the $(N+1) \times (N+1)$ coefficient matrix B, as presented in Ref. 1, are

$$b_{jm} = T_m(\eta_j), \quad j = 0, 1, \ldots, N \quad m = 0, 1, \ldots, N \tag{1d}$$

The vectors $\bar{g}(\xi) = [g_0^N(\xi), g_1^N(\xi), \ldots, g_N^N(\xi)]^T$ and $\bar{h}(\xi) = [h_0^N(\xi), h_1^N(\xi), \ldots, h_N^N(\xi)]^T$ have components

$$g_0^N(\xi) = \theta_2 \tag{1e}$$

$$g_j^N(\xi) = \theta_i + \frac{1}{\alpha^2}\sum_{m=0}^{N} a_m^N(\xi)T_m''(\eta_j) - \frac{(1-\omega)}{N_{cr}}\sum_{m=0}^{N}[b_m^N(\xi) - c_m^N(\xi)]T_m(\eta_j)$$

$$j = 1, 2, \ldots, N-1 \tag{1f}$$

$$g_N^N(\xi) = 1 \tag{1g}$$

while

$$h_j^N(\xi) = [g_j^N(\xi)]^4, \qquad j = 0, 1, \ldots, N \tag{1h}$$

The components of the $(N+1) \times (N+1)$ coefficient matrix A are

$$a_{jm} = T_m(\eta_j) - (\alpha\omega/2)A_m^{\alpha}(\eta_j)$$

$$j = 0, 1, \ldots, N, \quad m = 0, 1, \ldots, N \tag{1i}$$

The unknown time-varying components in $\bar{f}(\xi) = [f_0^N(\xi), f_1^N(\xi), \ldots, f_N^N(\xi)]^T$ are given as

$$f_j^N(\xi) = h(\eta_j, \xi) + \frac{\alpha(1-\omega)}{2}\sum_{m=0}^{N} b_m^N(\xi)A_m^{\alpha}(\eta_j)$$

$$j = 0, 1, \ldots, N \tag{1j}$$

Here, $T_m(\eta)$ represents the mth Chebyshev polynomial for the first kind, and η_j represents the jth collocation point as given in Ref. 1. The function $h(\eta_j, \xi)$ and constants $A_m^{\alpha}(\eta_j)$ are presented by Frankel.[1]

Though notationally not explicit, it is clear from viewing the definitions of the vector functions $\bar{f}(\xi)$, $\bar{g}(\xi)$, and $\bar{h}(\xi)$ that coupling exists among the unknown time-varying expansion coefficients of the cumulative variables. The physical variables can be reconstructed[1] once satisfactory numerical closure on the cumulative variables has been ordained.

The unknown expansion coefficients for $\Psi_3^N(\eta, \xi)$ are denoted by $\{c_m^N(\xi)\}_{m=0}^N$. These functions can be symbolically resolved in terms of the functions $\{a_m^N(\xi)\}_{m=0}^N$ and $\{b_m^N(\xi)\}_{m=0}^N$ using *Mathematica*™.[8] Formally, this is done through an inverse operation, i.e.,

$$\bar{c}(\xi) = A^{-1}\bar{f}(\xi), \qquad \xi \geq 0 \tag{2}$$

when $|A| \neq 0$. Thus, the components of the vector $\bar{c}(\xi)$ can be eliminated from $\bar{g}(\xi)$ and $\bar{h}(\xi)$. This, in effect, reduces the differential/algebraic matrix system shown in Eqs. (1a–1c) to a pure differential matrix system. Equations (1a) and (1b) constitute the starting point for the development of the implicit numerical method. This approach is taken due to the already noted stability constraints imposed on the initial value scheme as the order of the approximation is increased.[1] In Ref. 1, a fully explicit fifth-order Runge–Kutta method was used in solving the time-varying expansion coefficients $\{a_m^N(\xi)\}_{m=0}^N$ and $\{b_m^N(\xi)\}_{m=0}^N$. As noted,[1] the time steps required in an explicit method must decrease in size as N increases in order to ensure convergence. To overcome this

Received Oct. 28, 1994; revision received Dec. 17, 1994; accepted for publication Jan. 24, 1995.

*Associate Professor, Mechanical and Aerospace Engineering Department.

unfortunate "stiffening," an implicit method is presently offered.

To begin, an equivalent integral formulation of Eqs. (1a) and (1b), namely, by direct integration, is

$$\bar{a}(\xi) = \bar{a}(0) + \int_{\xi_0=0}^{\xi} B^{-1}\bar{g}(\xi_0)\, d\xi_0 \tag{3a}$$

$$\bar{b}(\xi) = \bar{b}(0) + \int_{\xi_0=0}^{\xi} B^{-1}\bar{h}(\xi_0)\, d\xi_0, \qquad \xi \geq 0 \tag{3b}$$

where B^{-1} has components of β_{jm}, $j = 0, 1, \ldots, N$ and $m = 0, 1, \ldots, N$. The product of $B^{-1}\bar{g}(\xi)$ is the vector given by

$$B^{-1}\bar{g}(\xi) = [\gamma_0^N(\xi), \gamma_1^N(\xi), \ldots, \gamma_N^N(\xi)]^T \tag{4a}$$

where

$$\gamma_i^N(\xi) = \sum_{j=0}^{N} \beta_{ij} g_j^N(\xi), \qquad i = 0, 1, \ldots, N \tag{4b}$$

Likewise, the product of $B^{-1}\bar{h}(\xi)$ is

$$B^{-1}\bar{h}(\xi) = [\nu_0^N(\xi), \nu_1^N(\xi), \ldots, \nu_N^N(\xi)]^T \tag{5a}$$

where

$$\nu_i^N(\xi) = \sum_{j=0}^{N} \beta_{ij} h_j^N(\xi) = \sum_{j=0}^{N} \beta_{ij} [g_j^N(\xi)]^4, \qquad i = 0, 1, \ldots, N \tag{5b}$$

With the aid of Eqs. (4) and (5), Eqs. (3a) and (3b) can be written as

$$a_i^N(\xi) = a_i^N(0) + \sum_{j=0}^{N} \beta_{ij} \int_{\xi_0=0}^{\xi} g_j^N(\xi_0)\, d\xi_0 \qquad i = 0, 1, \ldots, N \tag{6a}$$

$$b_i^N(\xi) = b_i^N(0) + \sum_{j=0}^{N} \beta_{ij} \int_{\xi_0=0}^{\xi} [g_j^N(\xi_0)]^4\, d\xi_0 \qquad i = 0, 1, \ldots, N, \quad \xi \geq 0 \tag{6b}$$

respectively.

At this point, a block-by-block method can be implemented using a Simpson's-based rule.[5-7] Block-by-block methods (implicit Runge–Kutta methods[5]) are self-starting and produce solutions for the unknown functions in blocks of time. Using the Simpson's base, the unknown variables in blocks con-

Table 1 Dimensionless temperature and radiative heat flux results at three spatial locations when $\xi = 0.05$, $N_{cr} = 0.1$, $\omega = 0.5$, $\alpha = 0.5$, and $\theta_i = \theta_2 = 0$ as obtained using different time steps $\Delta\xi$ and varying orders N

			$\theta_N(\eta, 0.05)$			$\frac{Q_N^r(\eta, 0.05)}{N_{cr}}$		
ω	N	$\Delta\xi$	$\eta = -0.5$	$\eta = 0$	$\eta = 0.5$	$\eta = -1$	$\eta = 0$	$\eta = 1$
0.5	4	0.01	0.4995	0.1798	0.0504	1.9355	1.3024	0.8338
		0.005	0.4996	0.1797	0.0504	1.9355	1.3025	0.8339
	6	0.01	0.4885	0.1778	0.0583	1.9344	1.3285	0.8318
		0.005	0.4888	0.1777	0.0584	1.9348	1.3284	0.8317
	8	0.01	0.4765	0.1767	0.0631	1.9089	1.3453	0.8376
		0.005	0.4895	0.1774	0.0587	1.9344	1.3287	0.8319
		0.0025	0.4893	0.1773	0.0587	1.9342	1.3289	0.8319

Table 2 Dimensionless temperature and radiative heat flux results at three spatial locations when $\xi = 0.05$, $N_{cr} = 0.1$, $\alpha = 0.5$, and $\theta_i = \theta_2 = 0$ as obtained using different time steps $\Delta\xi$ and varying orders N

		$\theta_N(\eta, 0.05)$			$\frac{Q_N^r(\eta, 0.05)}{N_{cr}}$		
N	$\Delta\xi$	$\eta = -0.5$	$\eta = 0$	$\eta = 0.5$	$\eta = -1$	$\eta = 0$	$\eta = 1$
			a) $\omega = 0$				
4	0.01	0.5241	0.2037	0.0643	2.1108	1.2617	0.6382
	0.005	0.5241	0.2036	0.0643	2.1107	1.2618	0.6382
6	0.01	0.5149	0.2029	0.0717	2.1144	1.3006	0.6338
	0.005	0.5152	0.2026	0.0718	2.1151	1.3006	0.6336
8	0.01	0.5031	0.2025	0.0773	2.0697	1.3316	0.6415
	0.005	0.5162	0.2024	0.0722	2.1139	1.3007	0.6340
	0.0025	0.5160	0.2023	0.0721	2.1135	1.3010	0.6341
			b) $\omega = 1$				
4	0.01	0.4408	0.1169	0.0089	1.3832	1.3806	1.3832
	0.005	0.4408	0.1169	0.0089	1.3832	1.3806	1.3832
6	0.01	0.4288	0.1142	0.0173	1.3835	1.3842	1.3835
	0.005	0.4289	0.1141	0.0175	1.3835	1.3842	1.3835
8	0.01	0.4170	0.1129	0.0217	1.3835	1.3833	1.3835
	0.005	0.4293	0.1139	0.0176	1.3835	1.3833	1.3835
	0.0025	0.4292	0.1139	0.0176	1.3835	1.3833	1.3835

taining two time steps are obtained. Excellent expositions on this rule are found in Refs. 5–7.

III. Results and Discussion

The numerical results presented in this section were obtained using a computer code written in *Mathematica*. It should be remarked that using a hybrid method that links the *Mathematica* code (used for setting up the matrix equations) to either a C or Fortran code (used for performing the iteration operations), would produce a highly effective computational procedure.

In the tables presented here, the underlined results correspond to actual collocation positions where the residual is forced to be zero.[1] For the sake of comparison with Frankel,[1] Table 1 presents results for the reconstructed physical variables of dimensionless temperature $\theta_N(\eta, \xi)$ and dimensionless radiative heat flux $Q_N^r(\eta, \xi)$ as both a function of the time step $\Delta\xi$ used in solving the expansion coefficients, and the order of the approximation N when the conduction/radiation number is $N_{cr} = 0.1$, the single-scattering albedo is $\omega = 0.5$, the dimensionless half width is $\alpha = 0.5$, and the dimensionless initial temperature and defined surface temperature at $\eta = 1$ are $\theta_i = \theta_2 = 0$, respectively. The results indicated in Table 1 correspond well with those of the explicitly obtained results reported by Frankel,[1] which in turn support the results described by other investigators.

In most reported studies, sparse numerical accounts have been documented for the extremal values of the single-scattering albedo, i.e., $\omega = 0, 1$. It should be noted that the numerical results presented in Table 2 were obtained without any modification to the general computer code developed for $\omega\varepsilon(0, 1)$. Table 2a presents numerical results for both the dimensionless temperature $\theta_N(\eta, \xi)$ and dimensionless radiative heat flux $Q_N^r(\eta, \xi)$ as both a function of the time step $\Delta\xi$ used in solving the expansion coefficients and the order of the approximation N when $N_{cr} = 0.1$, $\omega = 0$, $\alpha = 0.5$, and $\theta_i = \theta_2 = 0$. By comparing the results of Table 2a to the results of Table 1, it is clear that additional energy is available for absorption when the single-scattering albedo is decreased. This corresponds to an increase in the temperature distribution as indicated in Table 2a.

Table 2b presents tabulated dimensionless temperature and radiative heat flux results for the conservative case ($\omega = 1$) when $N_{cr} = 0.1$, $\alpha = 0.5$, and $\theta_i = \theta_2 = 0$ for various time step sizes $\Delta\xi$ and various orders of the approximation N. It is well-known that under this situation the radiative heat flux is constant as clearly seen in this table. Also, the resulting temperature distribution is in line with physical expectations in comparison with Tables 1 and 2a.

The mathematical formalism described by Frankel[1] and computationally refined here offers an alternative methodology for solving transient mixed-mode transport. Additional studies are underway generalizing the methodology to multidimensional geometries and various degrees of anisotropy.

Acknowledgment

This work was supported under Grant CCR-9320385 provided by the National Science Foundation.

References

[1]Frankel, J. I., "Cumulative Variable Formulation for Transient Conductive and Radiative Transport in Participating Media," *Journal of Thermophysics and Heat Transfer*, Vol. 9, No. 2, 1995, pp. 210–218.

[2]Frankel, J. I., "A New Orthogonal Collocation Integral Formulation to Transient Radiative Transport," *Boundary Element Technology IX*, Computational Mechanics Publication, Southampton, England, UK, 1994, pp. 51–58.

[3]Frankel, J. I., and Choudhury, S. R., "Some New Observations on the Classical Logistic Equation with Heredity," *Applied Mathematics and Computation*, Vol. 58, 1993, p. 275–308.

[4]Kumar, S., and Sloan, I. H., "A New Collocation-Type Method for Hammerstein Integral Equations," *Mathematics of Computation*, Vol. 48, No. 178, 1987, pp. 585–593.

[5]Linz, P., "Analytical and Numerical Methods for Volterra Equations," Society for Industrial and Applied Mathematics, Philadelphia, PA, 1985.

[6]Baker, C. T. H., *The Numerical Treatment of Integral Equations*, Clarendon, Oxford, England, UK, 1978.

[7]Delves, L. M., and Mohamad, J. L., *Computational Methods for Integral Equations*, Cambridge Univ. Press, Cambridge, England, UK, 1988.

[8]Wolfram, S., *Mathematica: A System for Doing Mathematics by Computer*, 2nd ed., Addison-Wesley, New York, 1991.

Towards a Theory of the Surface Conductance Coefficient in Solids

María L. Cancillo* and Juan J. Morales†
Universidad de Extremadura, Badajoz, Spain 06071

Nomenclature

c	=	specific heat
H	=	surface conductance coefficient
K	=	thermal conductivity
p	=	perimeter
sqdif	=	sum of the squares of the differences between the experimental data T_{ie} and the theoretical values T_{it}
T	=	temperature
T_0	=	constant temperature of the medium
t	=	time
w	=	cross section
x	=	coordinate in the axial direction
ρ	=	density

Introduction

THE surface conductance coefficient appears in a term of the differential equation for heat flow, which for the one-dimensional case takes the form:

$$\frac{\partial T}{\partial t} = \frac{K}{\rho c}\frac{\partial^2 T}{\partial x^2} - \frac{Hp}{\rho c w}(T - T_0) \tag{1}$$

Table 1 Parameters for the fit of the experimental steady-state data to the single exponential law of Eq. (2)

Cases	$T_1 \pm \sigma$	$-m \pm \sigma$	H, Eq. (3)	*sqdif*
I	75.2 ± 0.6	0.99 ± 0.05	2.35	17.37
II	68.1 ± 0.6	1.01 ± 0.05	2.44	15.12
III	65.0 ± 0.6	1.00 ± 0.06	2.39	13.13
IV	56.0 ± 0.5	0.97 ± 0.03	2.25	9.45
V	49.7 ± 0.4	0.97 ± 0.03	2.25	7.34
VI	43.0 ± 0.4	0.93 ± 0.03	2.07	5.18
VII	39.7 ± 0.4	0.88 ± 0.03	1.85	2.98
VIII	33.8 ± 0.4	0.92 ± 0.03	2.03	3.14
IX	30.2 ± 0.4	0.87 ± 0.03	1.81	1.63
X	23.5 ± 0.4	0.89 ± 0.04	1.90	1.47
XI	20.1 ± 0.4	0.82 ± 0.04	1.61	0.63
XII	13.9 ± 0.4	0.85 ± 0.04	1.65	0.33
Means	$\bar{m} = 0.92 \pm 0.06$		$\bar{H} = 2.03 \pm 0.12$	

Received Dec. 20, 1993; revision received Feb. 25, 1994; accepted for publication Feb. 25, 1994.

*Associated Lecturer, Department of Physics, Faculty of Sciences.
†Lecturer, Department of Physics, Faculty of Sciences.